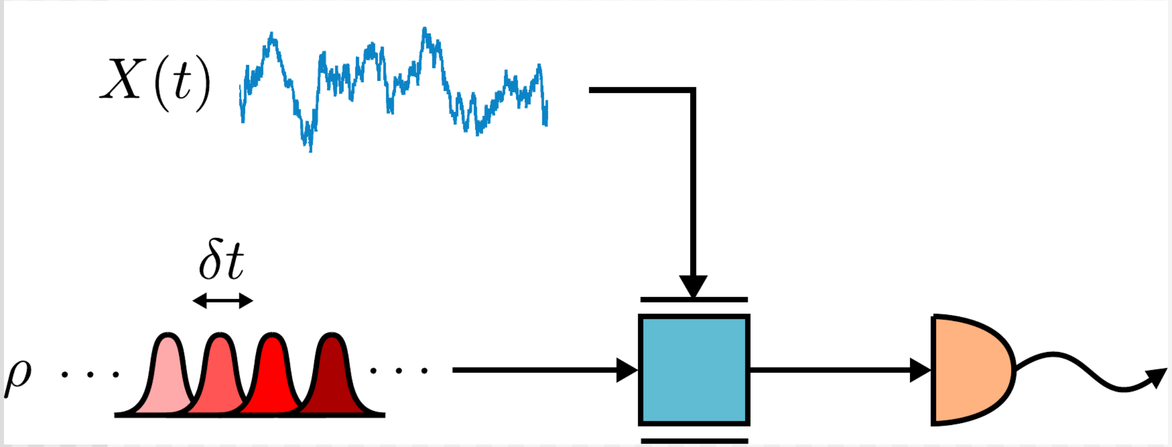
The Heisenberg limit for waveform estimation



Dominic Berry

Macquarie University

Mankei Tsang

National University of Singapore

Michael Hall

Howard Wiseman

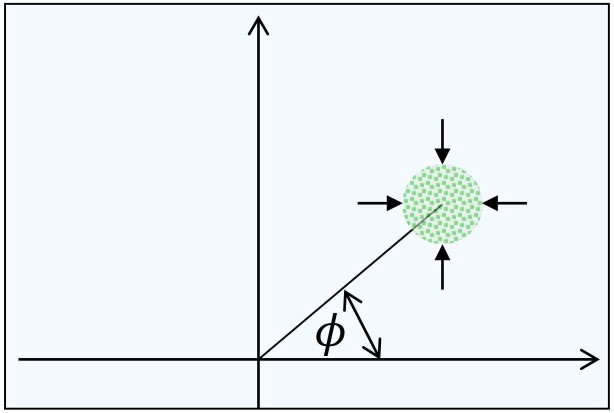
Griffith University

arXiv:1409.7877

The Heisenberg limit *vs* the standard quantum limit

The Standard Quantum Limit

- Coherent states.

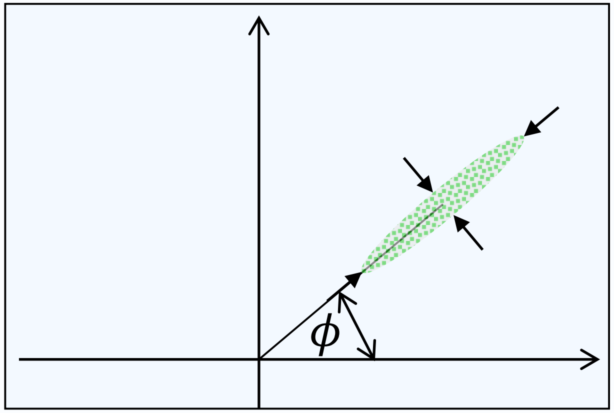


- Error scaling

$$\Delta\phi^2 \propto 1/N$$

The Heisenberg Limit

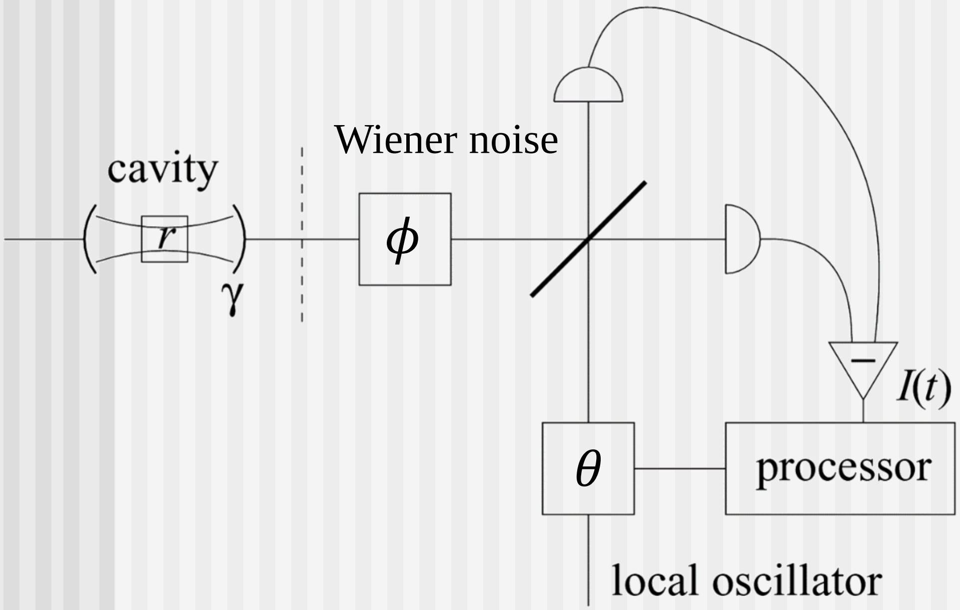
- Squeezed states.

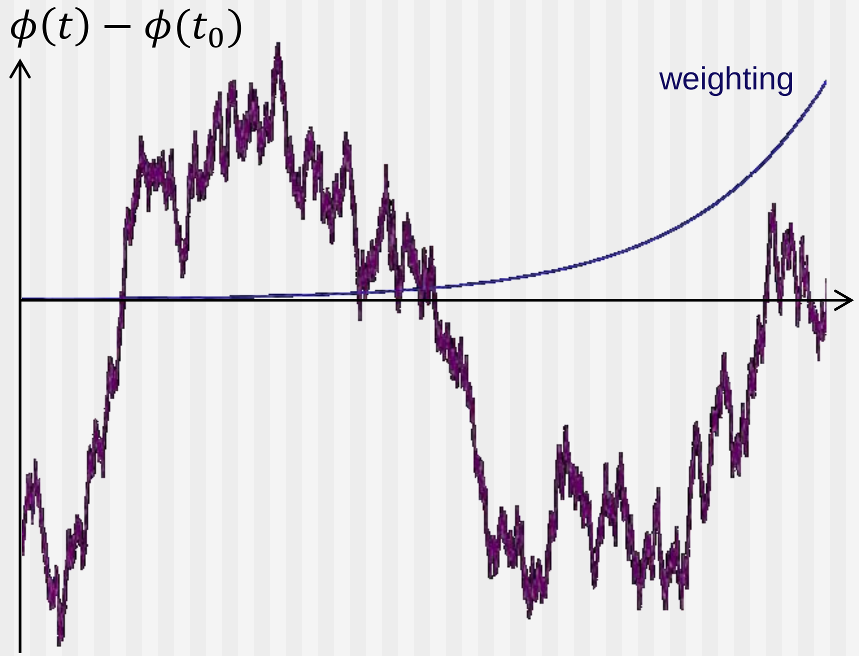


- Error scaling

$$\Delta\phi^2 \propto 1/N^2$$

Measurement of varying phase





Constant *vs* varying

- Coherent states:

$$\Delta\phi^2 \propto 1/N \quad vs \quad \Delta\phi^2 \propto 1/\sqrt{N}$$

- Squeezed states:

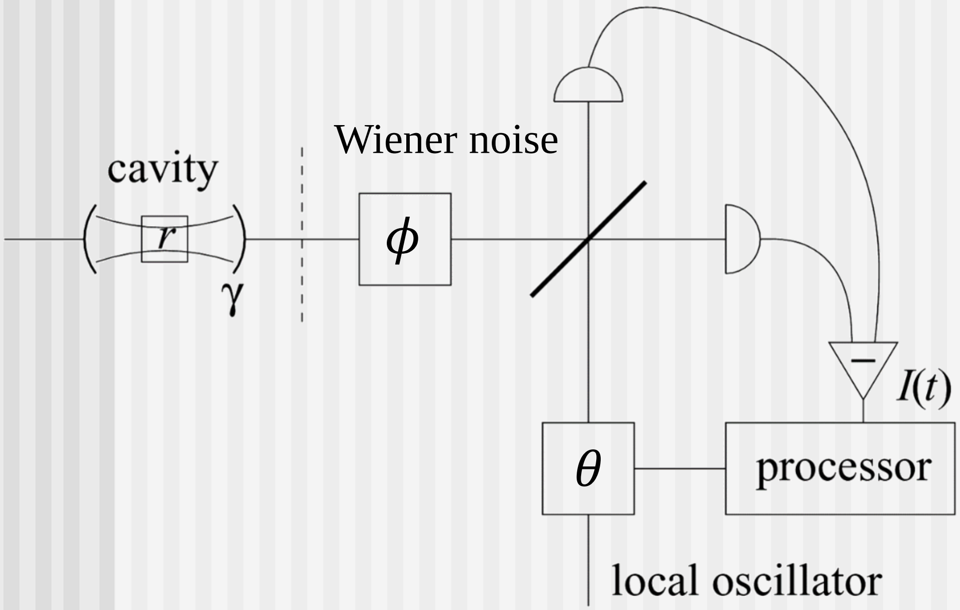
$$\Delta\phi^2 \propto e^{-2r}/N \quad vs \quad \Delta\phi^2 \propto e^{-r}/\sqrt{N}$$

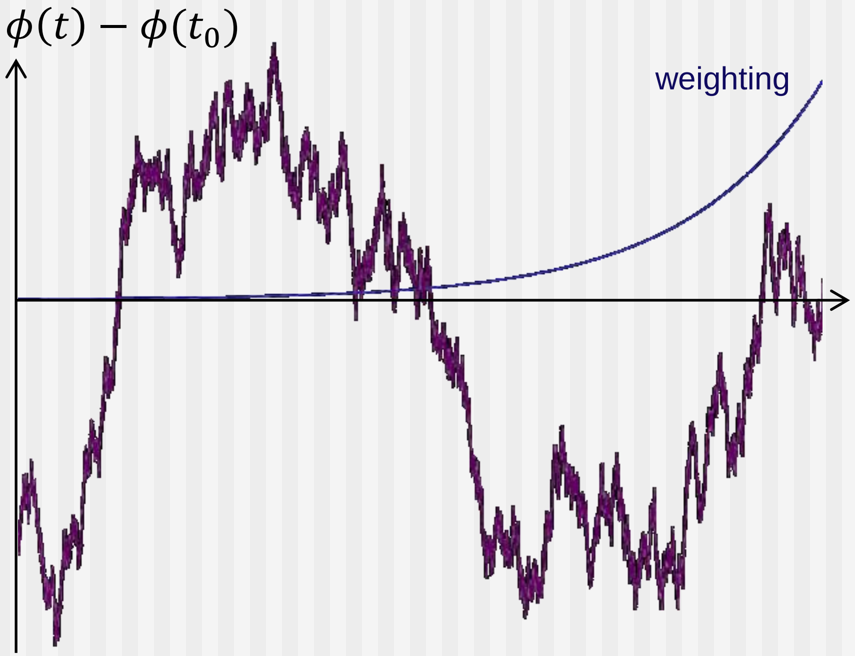
- Optimal states:

$$\Delta\phi^2 \propto 1/N^2 \quad vs \quad \Delta\phi^2 \propto 1/N?$$

D. W. Berry & H. M. Wiseman,
PRA **65**, 043803 (2002)
PRA **73**, 063824 (2006)
PRA **87**, 019901(E) (2013).

Measurement of varying phase





Constant *vs* varying

- Coherent states:

$$\Delta\phi^2 \propto 1/N \quad vs \quad \Delta\phi^2 \propto 1/\sqrt{N}$$

- Squeezed states:

$$\Delta\phi^2 \propto e^{-2r}/N \quad vs \quad \Delta\phi^2 \propto e^{-r}/\sqrt{N}$$

- Optimal states:

$$\Delta\phi^2 \propto 1/N^2 \quad vs \quad \Delta\phi^2 \propto 1/N^{2/3}$$

D. W. Berry & H. M. Wiseman,
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Types of phase correlations

- Signal correlations

$$\Sigma(t-t') = \langle[\phi(t)-\phi(t')]^2\rangle$$

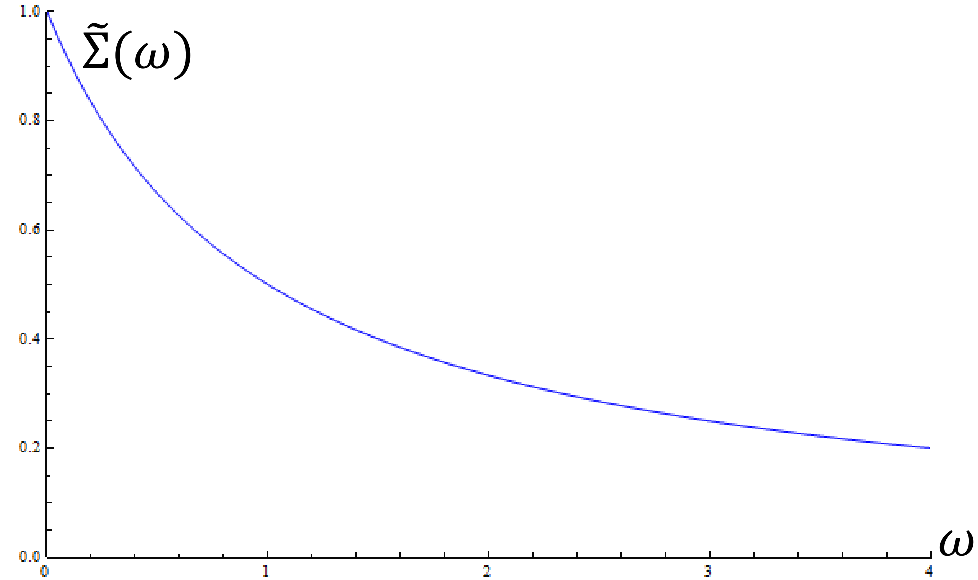
- Spectrum scaling

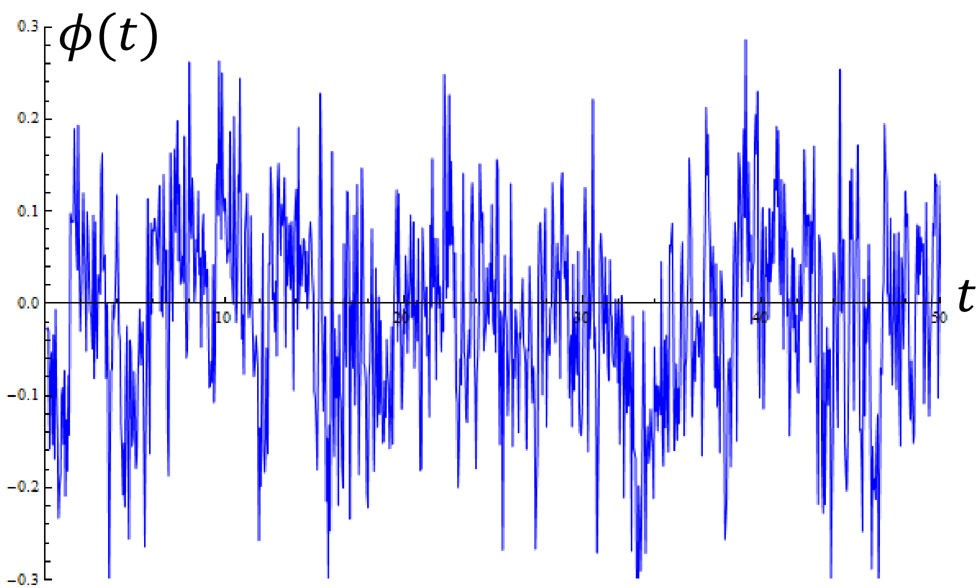
$$\tilde{\Sigma}(\omega) \propto \frac{1}{|\omega|^p}$$

- Exact spectrum

$$\tilde{\Sigma}(\omega) = \frac{\kappa^{p-1}}{|\omega|^p + \gamma^p}$$

- Statistics for **signal** are Gaussian and stationary.





Our result

Heisenberg limit for waveform estimation is

$$\Delta\phi^2 \geq \frac{c_Z(p)}{N^{2(p-1)/(p+1)}}$$

Measurements are possible with

$$\Delta\phi^2 \sim \frac{c_A(p)}{N^{2(p-1)/(p+1)}}$$

N is average flux divided by κ.

Standard quantum limit is

$$\Delta\phi^2 \propto \frac{1}{N^{(p-1)/p}}$$

- For $p = 2$

$\Delta\phi^2 \propto 1/N^{2/3}$ *vs*

$\Delta\phi^2 \propto 1/N^{1/2}$ SQL

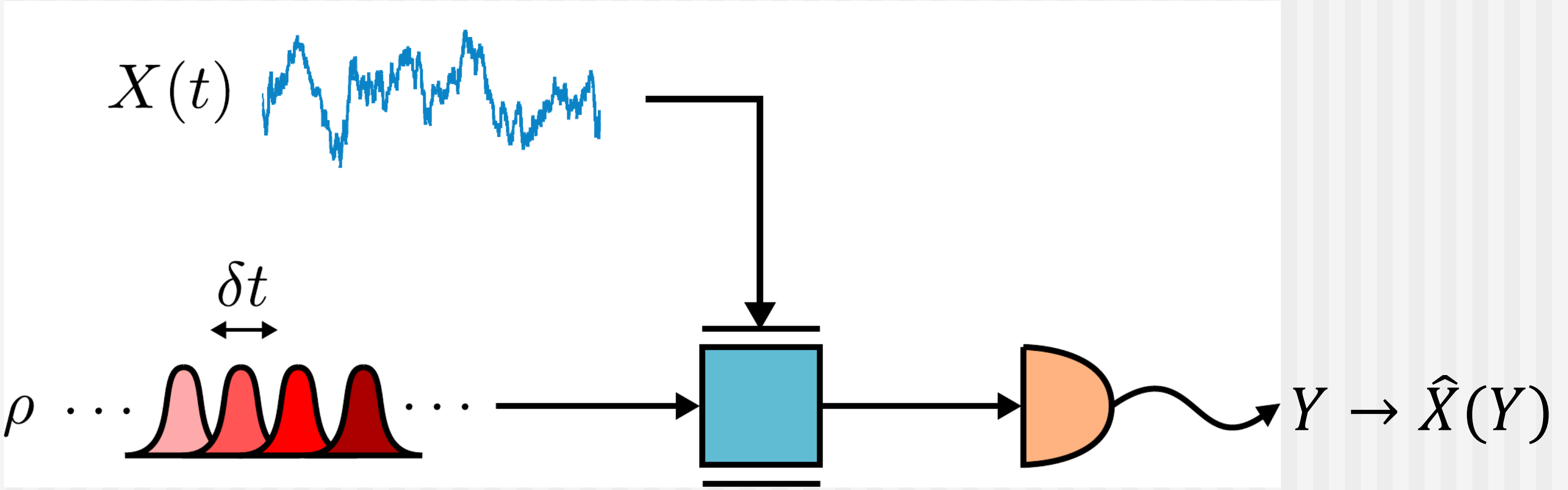
- For $p \to \infty$

$\Delta\phi^2 \propto 1/N^2$ *vs*

$\Delta\phi^2 \propto 1/N$ SQL

Bell-Ziv-Zakai bounds

- phases X
- measurement results Y
- estimates $\hat{X}(Y)$
- error $\epsilon = \hat{X}(Y) - X$
- We want mean-square error for estimation of phase X_k.



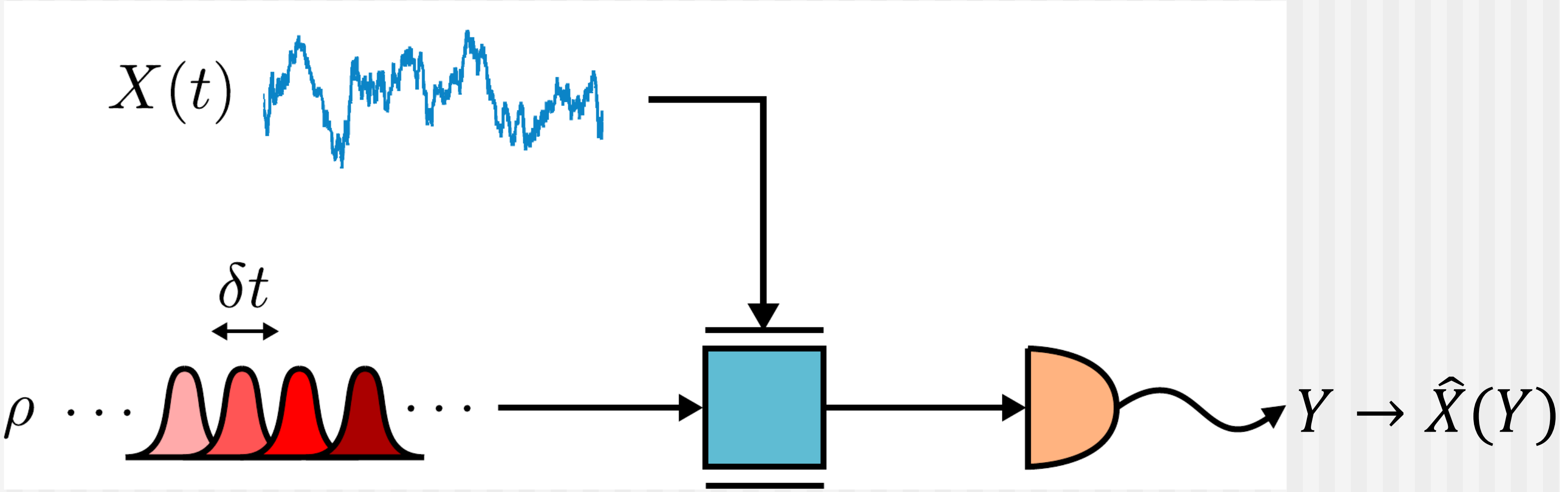
Bell-Ziv-Zakai bounds

$$\Delta\phi^2(t_0) = \frac{1}{2}\int_0^\infty d\tau\, \dot{D}\left(\frac{\tau}{2}\right) \Pr\left(|u^\top \epsilon| \geq \frac{\tau}{2}\right)$$

$$\Pr\left(|u^\top \epsilon| \geq \frac{\tau}{2}\right) \geq 2 \max_{v:u^\top v=1} \int dx\, \min[P_X(x), P_X(x+v\tau)] P_e(x, x+v\tau)$$

$$P_e(x, x+v\tau) \geq \frac{1}{2}\left[1 - \sqrt{1 - F(x, x+v\tau)}\right]$$

The **quantum Bell-Ziv-Zakai** bound!



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$$P_e(x, x+v\tau) \geq \frac{1}{2}\left[1 - \sqrt{1 - F(x, x+v\tau)}\right]$$

The **quantum Bell-Ziv-Zakai** bound!

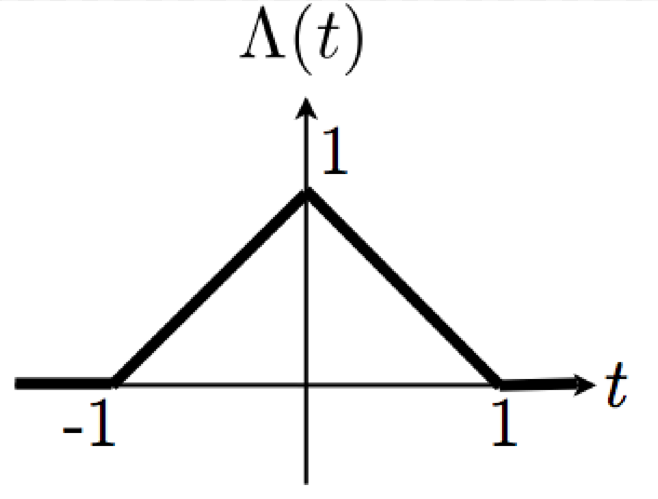
We need lower bounds on two things:

1. $F(x, x+v\tau)$ ← property of **state**
2. $\min[P_X(x), P_X(x+v\tau)]$ ← property of phase variation

Bound on F

We need lower bounds on two things:

1. $F(x, x + v\tau)$
2. $\min[P_X(x), P_X(x + v\tau)]$



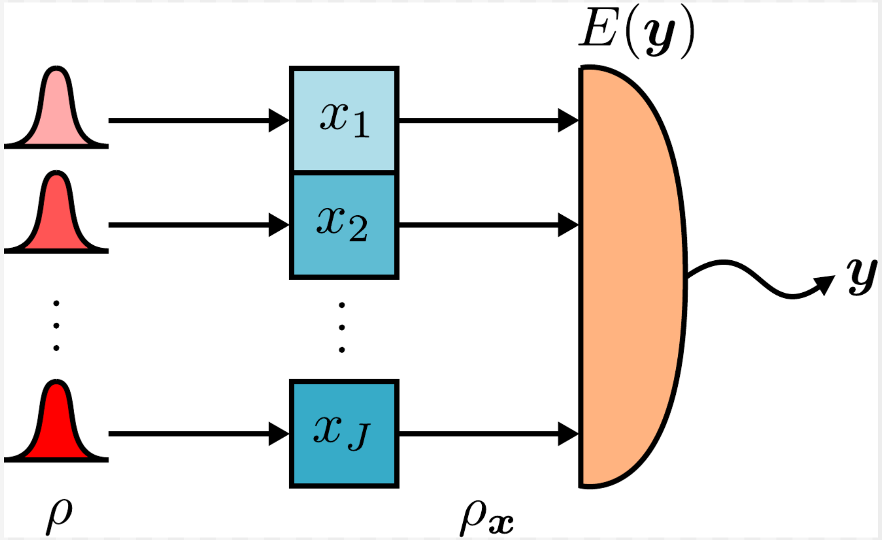
For phase measurement,

$$\rho_x = \exp(ix^\top n)\,\rho\,\exp(-ix^\top n)$$

We can show that (with $\lambda \approx 0.7246$)

$$F(x, x + v\tau) \geq \Lambda\left(\frac{\tau}{\tau_F}\right)$$

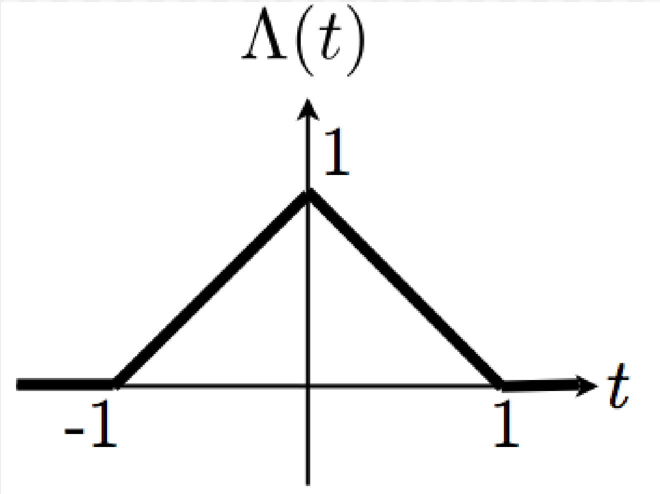
$$\tau_F = \frac{1}{2\lambda |v|^\top \langle n \rangle}$$



Bound on P_X

We need lower bounds on two things:

1. $F(x, x + v\tau)$
2. $\min[P_X(x), P_X(x + v\tau)]$



For phase measurement,

$$\rho_x = \exp(i x^\top n)\, \rho \exp(-i x^\top n)$$

We can show that (with $\lambda \approx 0.7246$)

$$F(x, x + v\tau) \geq \Lambda\left(\frac{\tau}{\tau_F}\right)$$

$$\tau_F = \frac{1}{2\lambda |v|^\top \langle n \rangle}$$

Correlations are multivariate Gaussian distribution with covariance Σ_0,

$$\int dx \min[P_X(x), P_X(x + v\tau)] \geq \Lambda\left(\frac{\tau}{\tau_0}\right)$$

$$\tau_0 = \sqrt{\frac{2\pi}{v^\top \Sigma_0^{-1} v}}$$

Net result

$$\Delta\phi^2(t_0) \geq \frac{1}{2}\int_0^\infty d\tau\, \tau\Lambda\left(\frac{\tau}{\tau_0}\right)\Lambda\left(\frac{\tau}{\tau_F}\right)$$

$$u^\top v = 1 \qquad \tau_F = \frac{1}{2\lambda|v|^\top\langle n\rangle} \qquad \tau_0 = \sqrt{\frac{2\pi}{v^\top\Sigma_0^{-1}v}}$$

Take:

$$v(t) = \mathrm{sinc}^2\left(\frac{t-t_0}{2T}\right)$$

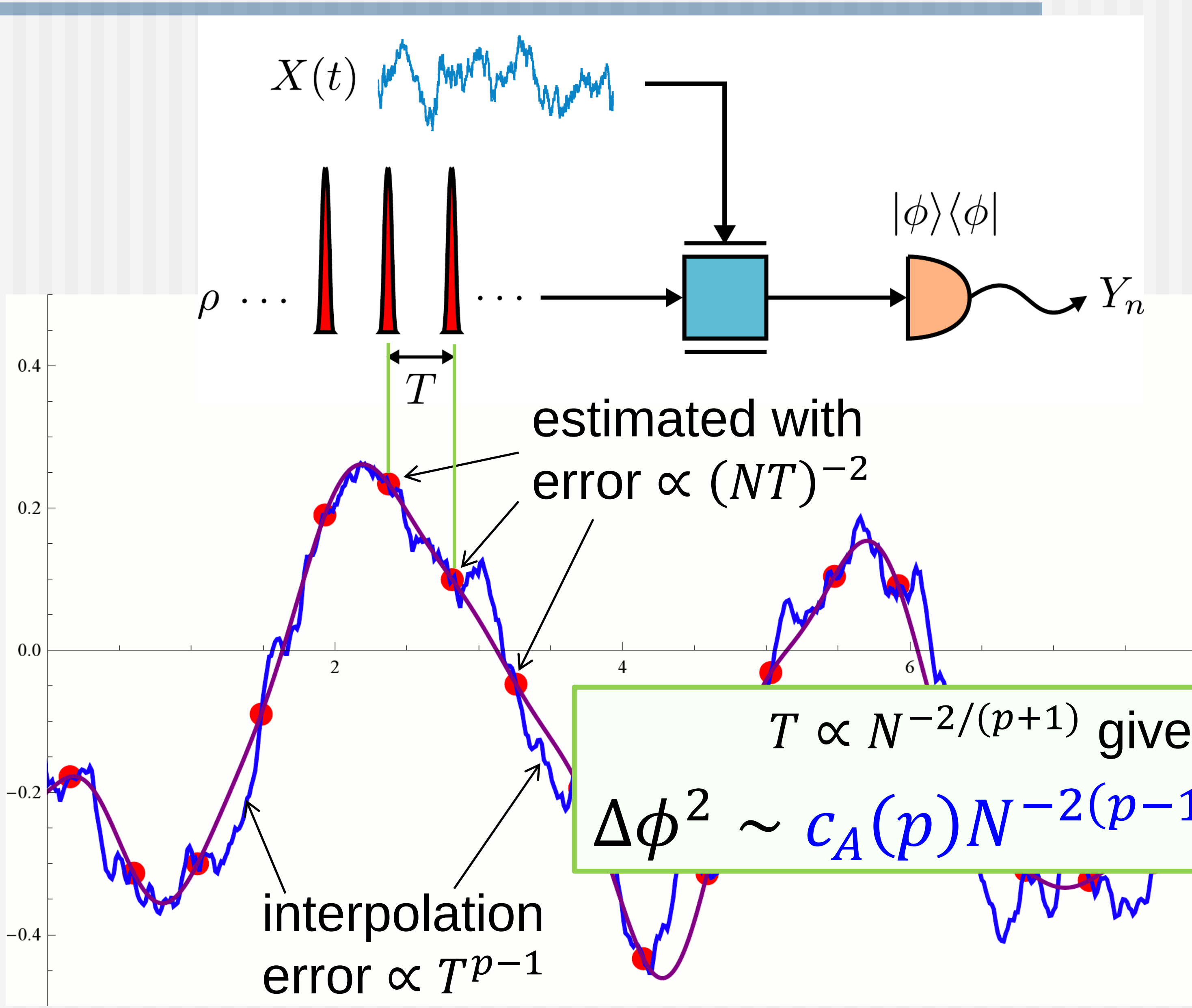
Gives

$$|v|^\top\langle n\rangle = 2\pi T N(t_0)$$
$$v^\top\Sigma_0^{-1}v \propto T^{1-p}$$

$T \propto N^{-2/(p+1)}$ gives

$$\Delta\phi^2 \geq c_z(p)N^{-2(p-1)/(p+1)}$$

Achieving the bound



$T \propto N^{-2/(p+1)}$ gives

$$\Delta\phi^2 \sim c_A(p) N^{-2(p-1)/(p+1)}$$

Conclusions

- We have proven a **Heisenberg limit** for waveform estimation, for phase variation with power-law correlations.

$$\tilde{\Sigma}(\omega) = \frac{\kappa^{p-1}}{|\omega|^p + \gamma^p} \quad \Rightarrow \quad \Delta\phi^2 \geq \frac{c_Z(p)}{N^{2(p-1)/(p+1)}}$$

- This shows that adaptive measurements proposed for squeezed states are optimal.

- This result appears as an application of the more general quantum Bell-Ziv-Zakai bound.

D. W. Berry, M. Tsang, M. J. W. Hall, H. M. Wiseman, arXiv:1409.7877

PhD opportunities at Macquarie University